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TOPOLOGICALLY DISTINCT CONJUGATE VARIETIES WITH FINITE FUNDAMENTAL GROUP

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I. INTRODUCTION AND STATEMENT OF RESULTS

§1. INTRODUCTION

LET V be a complex algebraic variety defined over an algebraic number field K. Any field automorphism σ of the complex numbers acts on the defining equations for V to produce a *conjugate variety* V^σ.

More formally, if V_K is a scheme over K and ε_1 an embedding $K \to \mathbb{C}$ we can form the pullback of the diagram

$$\begin{array}{ccc} & & V_K \\ & & \downarrow \\ \operatorname{Spec} \mathbb{C} & \xrightarrow{\varepsilon_1^*} & \operatorname{Spec} K \end{array}$$

to obtain a complex variety $V_{\mathbb{C}}^1$. Choosing a different embedding $\varepsilon_2 : K \to \mathbb{C}$ gives a different variety $V_{\mathbb{C}}^2$. The comparison of the topological invariants of conjugate varieties is, in essence, the study of which topological invariants are algebraic in character, i.e. can be computed from the point of view of abstract algebraic geometry.

In 1954 Serre proved [11] that the betti numbers of nonsingular projective varieties are algebraic in nature, and the more recent work of Artin and Mazur (see [1]) shows that the profinite completion of the homotopy type of any complex algebraic variety can be algebraically attained. Indeed, this last fact when applied to the grassmann varieties has far-reaching implications for geometric topology (see [15]). On the other hand, Serre has also demonstrated that conjugate varieties need not have the same homotopy type. He produces [12] non-singular projective varieties V and V^σ such that $\pi_1(V) \neq \pi_1(V^\sigma)$. Of course, the profinite completions $\pi_1(V)^\wedge$ and $\pi_1(V^\sigma)^\wedge$ must be isomorphic so that the groups are necessarily infinite.

In this paper we construct non-equivalent conjugate varieties having finite fundamental groups. Our first examples are based upon a modified version of a construction due to Godeaux and Serre which produces "lens varieties". These are built using projective representations in much the same way that lens spaces come from linear representations. The fact that the representation is not linear, however, adds a twist to the 2-stem of the lens variety's Postnikov tower, and the twist changes under conjugation in a computable way. The result is that for finite groups G satisfying a certain cohomological criterion (Condition A) we can find nonsingular projective varieties V and V^σ with fundamental group G such that V and V^σ are not homotopy-equivalent.

We also deal with conjugate vector bundles. For any finite group G we can "approximate" the classifying space BG by a quasi-projective variety W. Studying representations of G gives information about vector bundles over W, and we find that if G satisfies some representation condition (Condition B) there are bundles E and E^σ over W whose total spaces are non-diffeomorphic conjugate quasi-projective varieties. In some cases these varieties are not even homeomorphic.

Neither these methods nor those of Serre produce conjugate varieties with distinct *rational homotopy type*. In fact, it is presently unknown whether or not conjugate varieties are rationally homotopy equivalent, or even have isomorphic real cohomology rings.

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§2. NON-HOMOTOPY-EQUIVALENT VARIETIES

If G is a group then any automorphism A of G induces a transformation $A^*\colon H^*(G) \to H^*(G)$. We concentrate on the action of $\mathrm{Aut}(G)$ on $H^3(G, \mathbb{Z})$ and extend this action to the (formal) exterior algebra $\Lambda^* H^3(G, \mathbb{Z})$ by

$$A^*(x_1 \wedge \cdots \wedge x_d) = A^*x_1 \wedge \cdots \wedge A^*x_d.$$

We say that $\xi \in \Lambda^* H^3(G, \mathbb{Z})$ *can be transported to* $\eta \in \Lambda^* H^3(G, \mathbb{Z})$ *by an automorphism of* G if there exists $A \in \mathrm{Aut}(G)$ with $A^*\xi = \eta$. Now consider the following condition on a group G, an integer l and cohomology classes $x_1, \ldots, x_d$ in $H^3(G, \mathbb{Z})$:

Condition A. G is a finite group of order k with l relatively prime to k and $x_1 \wedge \cdots \wedge x_d$ cannot be transported to $\pm l x_1 \wedge \cdots \wedge l x_d$ by an automorphism of G.

Then we have

THEOREM A. *Let $G, l, x_1, \ldots, x_d$ satisfy Condition A and let ω be a primitive kth root of unity. Then there exists a nonsingular projective algebraic variety V defined over $\mathbb{Q}(\omega)$ such that*

(i) $\pi_1(V) = G$.
(ii) *If σ is any galois automorphism with $\sigma(\omega) = \omega^l$ then V and V^σ are not homotopy equivalent.*

We prove this theorem in Part II and also give some examples.

§3. NON-DIFFEOMORPHIC VARIETIES

Let G be a group and A an automorphism of G. Then A acts on the representation ring of G, $\alpha \to A^*\alpha$. If G is a finite group of order k any complex representation of G is equivalent to one defined over $\mathbb{Q}(\omega)$ where ω is a kth root of unity [4]. Thus any galois automorphism σ of $\mathbb{Q}(\omega)$ also acts on the representation ring, $\alpha \to \alpha^\sigma$. The condition on α, σ and G which will allow us to build non-diffeomorphic varieties is:

Condition B. There is no automorphism A of G such that $A^*(\alpha \oplus \bar{\alpha}) = (\alpha \oplus \bar{\alpha})^\sigma$ as elements of the complex representation ring $R(G)$. (Here $\bar{\alpha}$ denotes the complex conjugate of α.)

In Chapter III we will show how to construct, for any group G and any complex representation of G a sequence of nonsingular quasi-projective varieties $V_n(\alpha)$ defined over $\mathbb{Q}(\omega)$ with $\pi_1(V_n(\alpha)) = G$. The main result of Chapter III is:

THEOREM B. *Let G be a p-group and α, σ chosen to satisfy Condition B. Then, for some n, $V_n(\alpha)$ and $V_n(\alpha^\sigma)$ are conjugate varieties which are homotopy equivalent but not diffeomorphic.*

Remark. In III §3 we use results of Adams and Sullivan to show that, for p in a certain class of primes, the varieties $V_n(\alpha)$ and $V_n(\alpha^\sigma)$ may be taken to be non-homeomorphic.

II. PROOF OF THEOREM A

§1. A CERTAIN 2-STAGE POSTNIKOV TOWER

Given a finite group G and classes $x_1, \ldots, x_d \in H^3(G, \mathbb{Z})$ let $E(x_1, \ldots, x_d)$ be the 2-stage Postnikov tower with $\pi_1 = G$, $\pi_2 = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ (d-times), trivial action of π_1 on π_2, and k-invariant $(x_1, \ldots, x_d)$. Then $E(x_1, \ldots, x_d)$ and $E(y_1, \ldots, y_d)$ are homotopy equivalent iff there is an automorphism A of G and a "coefficient automorphism" M of $\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ such that $(A^*x_1, \ldots, A^*x_d) = M(y_1, \ldots, y_d)$ in $\bigoplus_{i=1}^{d} H^3(G, \mathbb{Z})$. Since M is nothing more than a $d \times d$ unimodular matrix this implies that

$$A^*x_1 \wedge \cdots \wedge A^*x_d = \pm y_1 \wedge \cdots \wedge y_d \quad \text{in} \quad \Lambda^d H^3(G, \mathbb{Z}).$$

PROPOSITION II.1. *Let G be a finite group of order k and $x_1, \ldots, x_d$ any classes in $H^3(G, \mathbb{Z})$. Then for any integer n we can construct a non-singular projective algebraic variety V such that*

(i) *V is defined over $\mathbb{Q}(\omega)$ where ω is a primitive kth root of unity.*

(ii) *V is n-homotopy equivalent* (i.e. *homotopy equivalent up through the n-skeleton*) *to $E(x_1, \ldots, x_d)$.*

(iii) *Let σ be a galois automorphism with $\sigma(\omega) = \omega^l$. Then V^σ is n-homotopy-equivalent to $E(lx_1, \ldots, lx_d)$.*

We will prove this proposition in §II.4. Theorem A then follows immediately by taking $n \geq 3$.

§2. PROJECTIVE BUNDLES

We use the theory of projective bundles to obtain a more concrete hold on the Postnikov systems considered above.

Let $GL(n+1)$ denote the complex general linear group. If we divide the group by its center we obtain the n-dimensional projective linear group

$$PGL(n) = GL(n+1)/\mathbb{C}^*.$$

By a $\mathbb{P}^n$-*bundle* over a space X we mean a fiber bundle with fiber complex projective n-space and structure group $PGL(n)$. Such bundles are classified by homotopy classes of maps $X \to BPGL(n)$. The exact sequence

$$\mathbb{C}^* \to GL(n+1) \to PGL(n)$$

gives a corresponding fibration of classifying spaces

$$K(\mathbb{Z}, 2) \to BGL(n+1) \to BPGL(n).$$

Hence the obstruction to "linearizing a projective bundle", i.e. lifting the map $X \to BPGL(n)$ to $BGL(n+1)$ is given by a class in $H^3(X, \mathbb{Z})$.

LEMMA II.2. *Let $\mathbb{P}^n \to E \to BG$ be a projective bundle over BG with G a finite group. Then E is 2n-equivalent to the Postnikov tower*

$$\begin{tikzcd} K(\mathbb{Z}, 2) \arrow[r] & E_2 \arrow[d] & \\ & K(G, 1) \arrow[r, "k"] & K(\mathbb{Z}, 3) \end{tikzcd}$$

where the action of π_1 on π_2 is trivial and the k-invariant is (up to sign) the obstruction to linearizing the bundle.

Proof. The homotopy groups are correct and the simple connectivity of $BPGL(n)$ implies that the action of π_1 is trivial. The identification of the k-invariant follows from the work of Kundert [9] or Hsiang [7] which proves that the obstruction to linearizing is the same as the first obstruction to cross-section. Indeed, this fact can be seen directly by exhibiting an inclusion of fibrations

$$\begin{tikzcd} \mathbb{P}^n \arrow[r] \arrow[d] & EPGL(n) \arrow[d] \arrow[dr] & \\ & & BPGL(n) \\ K(\mathbb{Z},2) \arrow[r] & BGL(n+1) \arrow[ur] & \end{tikzcd}$$

where the top row is the universal $\mathbb{P}^n$-bundle.

§3. PROJECTIVE REPRESENTATIONS

We relate projective bundles over BG to projective representations of G.

By an *n-dimensional projective representation* of G we mean a homomorphism $\rho\colon G \to PGL(n)$. Two such representations are said to be projectively equivalent if they differ by an automorphism of $PGL(n)$. Given a projective representation, we may ask whether it is the projectivization of a linear representation, i.e. whether there exists a lifting

$$\begin{tikzcd} & GL(n+1) \arrow[d] \\ G \arrow[ur, dashed] \arrow[r, "\rho"] & PGL(n) \end{tikzcd}$$

The solution to this problem was given by Schur [10]. For each $x \in G$ choose $\tilde{\rho}(x) \in GL(n+1)$ which projects to $\rho(x)$. Then for any pair $x, y \in G$ we have

$$\tilde{\rho}(xy) = \alpha(x, y)\tilde{\rho}(x)\tilde{\rho}(y) \quad \text{for some} \quad \alpha(x, y) \in \mathbb{C}^*.$$

The function $\alpha\colon G \times G \to \mathbb{C}^*$ is called the *factor set* for $\tilde{\rho}$ and the relation $\tilde{\rho}((xy)z) = \tilde{\rho}(x(yz))$ yields the cocycle condition

$$\alpha(xy, z)\alpha(x, y) = \alpha(x, yz)\alpha(y, z) \qquad \forall x, y, z \in G$$

so that α represents a cohomology class in $H^2(G, \mathbb{C}^*)$. A different lifting $\tilde{\rho}'(x)$ of $\rho(x)$, gives a new factor set $\alpha'\colon G \times G \to \mathbb{C}^*$. But for each $x \in G$, $\tilde{\rho}'(x)$ is some non-zero multiple of $\tilde{\rho}(x)$, say $\tilde{\rho}'(x) = k(x)\tilde{\rho}(x)$.

Thus

$$\alpha'(x, y) = k(xy)k^{-1}(x)k^{-1}(y)\alpha(x, y)$$

so that α and α' determine the same cohomology class $s(\rho) \in H^2(G, C^*)$. This class is called the *Schur class* of the representation ρ and Schur's result is that ρ is projectively equivalent to a representation that lifts to $GL(n+1)$ iff $s(\rho) = 0$ in $H^2(G, \mathbb{C}^*)$.

Let EG be a free contractible G-space and let $\rho_1, \ldots, \rho_d$ be projective representations of G of dimensions $n_1, \ldots, n_d$. Then form

$$E(\rho_1, \ldots, \rho_d) = (EG \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_d})/G$$

which is a $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_d}$-bundle over BG. The homotopy-type of this space is determined by:

LEMMA II.3. *Let n be the minimum of the n_i. Then $E(\rho_1, \ldots, \rho_d)$ is $2n$-equivalent to the Postnikov tower $E(x_1, \ldots, x_d)$ where $x_i = \beta s(\rho_i)$. Here $s(\rho_i) \in H^2(G, \mathbb{C}^*)$ is the Schur class of ρ_i and β is the Bockstein homomorphism associated to the coefficient sequence*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\pi i} \mathbb{C} \xrightarrow{\exp} \mathbb{C}^* \longrightarrow 1.$$

Proof. $E(\rho_1, \ldots, \rho_d)$ is the pullback of $E(\rho_1) \times \cdots \times E(\rho_d)$ via the diagonal map $BG \to BG \times \cdots \times BG$. So we need to show that, for each i, the k-invariant of $E(\rho_i)$ is $\beta s(\rho_i)$.

Consider the diagram of topological groups

$$\begin{tikzcd}
 & & G \arrow[d, "\rho"] \\
\mathbb{C}_d^* \arrow[r] \arrow[d, "\gamma"] & GL(n+1)_d \arrow[r] \arrow[d] & PGL(n)_d \arrow[d] \\
\mathbb{C}^* \arrow[r] & GL(n+1) \arrow[r] & PGL(n)
\end{tikzcd}$$

where the groups in the top row are taken with the discrete topology and the vertical maps are the maps from discrete to continuous topology. The corresponding diagram of classifying spaces is

$$\begin{tikzcd}
 & & BG \arrow[d, "B\rho"] & & \\
K(\mathbb{C}^*, 1) \arrow[r] \arrow[d, "B\gamma"] & K(GL(n+1), 1) \arrow[r] \arrow[d] & K(PGL(n), 1) \arrow[r] \arrow[d] & K(\mathbb{C}^*, 2) \arrow[r] \arrow[d, "BB\gamma"] & {} \\
K(\mathbb{Z}, 2) \arrow[r] & BGL(n+1) \arrow[r] & BPGL(n) \arrow[r] & K(\mathbb{Z}, 3) \arrow[r] & {}.
\end{tikzcd}$$

The obstruction to linearizing is the composition

$$BG \to BPGL(n) \to K(\mathbb{Z}, 3).$$

The Schur class is the composition

$$BG \to K(PGL(n), 1) \to K(\mathbb{C}^*, 2).$$

So to prove the lemma we need only identify $BB\gamma: K(\mathbb{C}^*, 2) \to K(\mathbb{Z}, 3)$ with the Bockstein of the coefficient sequence.

To do this we take the diagram

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "2\pi i"] \arrow[d] & \mathbb{C}_d \arrow[r, "\exp"] \arrow[d] & \mathbb{C}_d^* \arrow[d, "\gamma"] \\
\mathbb{Z} \arrow[r, "2\pi i"] & \mathbb{C} \arrow[r, "\exp"] & \mathbb{C}^*
\end{tikzcd}$$

(where the top row is again taken with the discrete topology) and examine the corresponding diagram of classifying spaces

$$\begin{tikzcd}
K(\mathbb{Z}, 1) \arrow[r] \arrow[d, "\mathrm{id}"] & K(\mathbb{C}, 1) \arrow[r] \arrow[d] & K(\mathbb{C}^*, 1) \arrow[r, "\beta"] \arrow[d, "B\gamma"] & K(\mathbb{Z}, 2) \arrow[r] \arrow[d, "\mathrm{id}"] & {} \\
K(\mathbb{Z}, 1) \arrow[r] & * \arrow[r] & K(\mathbb{Z}, 2) \arrow[r] & K(\mathbb{Z}, 2) \arrow[r] & {}.
\end{tikzcd}$$

Then the Bockstein β in the top row is the one we are interested in and the "Bockstein" in the bottom row is the identity, since $\mathbb{C} \xrightarrow{\exp} \mathbb{C}^*$ is the universal covering which provides the identification of $\mathbb{C}^*$ with $K(\mathbb{Z}, 1)$.

§4. CONSTRUCTING THE LENS VARIETY

Let G be a finite group of order k, n a positive integer, and $x_1, \ldots, x_d$ classes in $H^3(G, \mathbb{Z})$. We will construct the variety V advertised in Proposition II.1. Consider first the case of only one cohomology class. To a class x in $H^3(G, \mathbb{Z})$ we associate a variety X by the following procedure.

Step 1. Construct an action: Since G is finite $H^2(G, \mathbb{C}) = 0$ and so the Bockstein $\beta: H^2(G, \mathbb{C}^*) \to H^3(G, \mathbb{Z})$ is an isomorphism. Thus we can choose $y \in H^2(G, \mathbb{C}^*)$ with $\beta y = x$. Moreover, we can choose a cocycle representative α for y which takes on values in $\mathbb{Q}(\omega)$ where ω is a kth root of unity.

Associate to $\alpha: G \times G \to \mathbb{C}^*$ (following Schur [10]) a modification of the regular representation for G given by: G acts on the complex vector space with basis $\{v_g\}_{g \in G}$ via $g \to T_g$ where

$$T_g(v_h) = \alpha(g, h)v_{gh}.$$

Then

$$\begin{aligned} T_f\, T_g\, v_h &= \alpha(g, h)T_f\, v_{gh} \\ &= \alpha(g, h)\alpha(f, gh)v_{fgh} \\ &= \alpha(f, g)\alpha(fg, h)v_{fgh} \quad \text{by the cocycle condition for } \alpha \\ &= \alpha(f, g)T_{fg}\, v_h \end{aligned}$$

which shows that T represents a projective representation with factor set α. Now consider the map $\tilde{\rho}: G \to GL(k(n+2))$ given by

$$\tilde{\rho}: g \to T_g \oplus T_g \oplus \cdots \oplus T_g \quad (n+2) \text{ times}.$$

Then $\tilde{\rho}$ also satisfies $\tilde{\rho}(fg) = \alpha(f, g)\tilde{\rho}(f)\tilde{\rho}(g)$ and thus represents a projective representation $\rho: G \to PGL(k(n+2) - 1)$ with factor set α. For each $g \in G$ the fixed point set of $\rho(g)$ acting on $P = \mathbb{P}^{k(n+2)-1}$ is a subvariety of codimension at least $n + 2$. Hence the G-action on P is free off a subvariety of codimension at least $n + 2$.

Step 2. Construct a variety: Following Godeaux and Serre [13] we choose a homogeneous basis of degree D polynomials $f_0, \ldots, f_N$ for the G-invariant polynomials on P. This provides an embedding of P/G as a variety in $\mathbb{P}^N$ and also a covering map $f: P \to P/G$. Since the singular set of P/G has codimension at least $n + 2$ we can choose a linear subspace L in $\mathbb{P}^N$ to cut out a non-singular $(n+1)$-dimensional section of P/G. Then $x = f^{-1}((L \cap P)/G)$ is an $(n+1)$-dimensional non-singular subvariety of P on which the G-action is free. Since the polynomials $\{f_i\}$ provide the coordinates in $\mathbb{P}^N$, L is cut out in $\mathbb{P}^N$ by linear combinations $\{g_j\}$ of the $\{f_i\}$. Then $X \subset P$ is cut out by these same $\{g_i\}$ viewed as polynomials in the coordinates of P. One can show [13] that the ideal (g_j) is actually the ideal of $X \subset P$ and hence that X is a complete intersection of hypersurfaces of degree D. Notice that if we choose the $\{g_j\}$ to be rational linear combinations of the $\{f_i\}$ then X is defined over $\mathbb{Q}(\omega)$.

Now we form the variety X_i for each cohomology class x_i and set $V = (X_1 \times X_2 \times \cdots \times X_d)/G$. We verify that V satisfies the three conditions of II.1:

(i) Each X_i is non-singular with a free G-action defined over $\mathbb{Q}(\omega)$. Hence V is non-singular and defined over $\mathbb{Q}(\omega)$.

(ii) Each X_i is a complex $(n+1)$-dimensional complete intersection of degree D hypersurfaces and so the Lefschetz theorem implies (as in [3]) that the inclusion $X_i \hookrightarrow P$ is an n-equivalence. Thus if EG is a free contractible G-space

$$EG \times X_1 \times \cdots \times X_d \hookrightarrow EG \times P \times \cdots \times P$$

is an n-equivalence so the quotient map

$$(EG \times X_1 \times \cdots \times X_d)/G \hookrightarrow (EG \times P \times \cdots \times P)/G$$

is also an n-equivalence. The action of G on X_i in the above equation is free, which implies that the space on the left fibers over V with fiber EG and therefore is homotopy equivalent to V. The space on the right is by definition, $E(\rho_1, \ldots, \rho_d)$ as defined in §3. Since the representations were chosen with the property $x_i = \beta s(\rho_i)$ for each i, Lemma II.3 shows that V is n-equivalent to the tower $E(x_1, \ldots, x_d)$.

(iii) Let $\sigma \in \mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})$ send ω to ω^l. For each ρ_i we consider the conjugate representation ρ_i^σ defined by $\rho_i^\sigma(t) = \sigma\rho_i(t)$ for $t \in P$. This makes good sense since the ρ_i are defined over $\mathbb{Q}(\omega)$. To obtain the invariant polynomials for the G-action defined by ρ^σ we merely conjugate the invariant polynomials for the G-action defined by ρ. Then following through step 2 of the construction using ρ_i^σ instead of ρ_i produces the conjugate V^σ of V which is n-equivalent to $E(\rho_1^\sigma, \rho_2^\sigma, \ldots, \rho_d^\sigma)$.

To compute the homotopy type we need to know the Schur class of ρ_i^σ. Notice what happens to the factor set under conjugation:

if
$$\rho(xy) = \alpha(x, y)\rho(x)\rho(y)$$
then
$$\rho^\sigma(xy) = \sigma[\alpha(x, y)\rho(x)\rho(y)] = \sigma\alpha(x, y)\rho^\sigma(x)\rho^\sigma(y).$$

So if $\alpha_i\colon G \times G \to \mathbb{Q}(\omega)$ represents the Schur class $s(\rho_i)$, the composition

$$G \times G \xrightarrow{\alpha_i} \mathbb{Q}(\omega) \xrightarrow{\sigma} \mathbb{Q}(\omega)$$

represents the Schur class of ρ_i^σ. Thus σ acting as coefficient homomorphism

$$\sigma\colon H^2(G, \mathbb{Q}(\omega)) \to H^2(G, \mathbb{Q}(\omega)) \quad \text{sends} \quad s(\rho_i) \quad \text{to} \quad s(\rho_i^\sigma).$$

Since σ extends to an l-fold covering $\mathbb{C}^* \to \mathbb{C}^*$ it is transformed via the coefficient sequence

$$0 \longrightarrow Z \xrightarrow{2\pi i} \mathbb{C} \xrightarrow{\exp} \mathbb{C}^* \longrightarrow 1$$

into multiplication by l on $H^3(G, \mathbb{Z})$ which shows that $\beta s(\rho_i^\sigma) = l\beta s(\rho_i)$. Therefore by II.3, $E(\rho_1^\sigma, \ldots, \rho_d^\sigma)$ is (up to dimension nk) equivalent to the Postnikov tower $E(lx_1, \ldots, lx_d)$.

§5. EXAMPLE

In this section we describe a family of nilpotent p-groups satisfying Condition A.

Let G be the "mod p Heisenberg group" given by generators and relations as

$$x^p = y^p = z^p = 1 \qquad [x, z] = 1$$
$$[x, y] = z \qquad [y, z] = 1.$$

G is a central extension of $\mathbb{Z}/p$ (generated by z) by $\mathbb{Z}/p \oplus \mathbb{Z}/p$ (generated by the images of x and y).

$$0 \to \underset{z}{\mathbb{Z}/p} \to G \to \underset{x}{\mathbb{Z}/p} \oplus \underset{y}{\mathbb{Z}/p} \to 0.$$

LEMMA II.5. *For G the* mod p *Heisenberg group there exist cohomology classes s, $t \in H^3(G, \mathbb{Z})$ such that G, l, s, t satisfy Condition A where l is any integer such that l^2 is not a cube* mod p. (Such integers l always exist whenever $p \equiv 1 \pmod 3$.)

Proof. To avoid notational confusion, we write G as the extension

$$1 \to C \to G \to W \to 0$$

where $C = \mathbb{Z}/p$, $W = \mathbb{Z}/p \oplus \mathbb{Z}/p$.

We consider the Hochschild–Serre spectral sequence for the mod p cohomology of G

$$H^p(W; H^q(C; \mathbb{Z}/p)) \Rightarrow H^*(G; \mathbb{Z}/p).$$

Notice that the action of W on $H^*(C; \mathbb{Z}/p)$ is trivial since G is a central extension.

Now $H^*(C; \mathbb{Z}/p)$ is an algebra generated by $\tilde{z}$ and $\beta\tilde{z}$ where $\tilde{z} \in H^1(C; \mathbb{Z}/p)$ corresponds to $z \in C$ and β is the Bockstein operation in mod p cohomology. Similarly $H^*(W; \mathbb{Z}/p)$ is generated by classes $\tilde{x}$, $\tilde{y}$, $\beta\tilde{x}$, $\beta\tilde{y}$.

Then the first few pieces in the E_2 term of the spectral sequence look like

$$\begin{array}{lll}
E^{02}: \beta\tilde{z} & & \\
E^{01}: \tilde{z} & E^{11}: \begin{matrix}\tilde{x}\otimes\tilde{z}\\ \tilde{y}\otimes\tilde{z}\end{matrix} & \\
 & E^{10}: \begin{matrix}\tilde{x}\\ \tilde{y}\end{matrix} & E^{20}: \begin{matrix}\tilde{x}\tilde{y}\\ \beta\tilde{x}\\ \tilde{y}.\end{matrix}
\end{array}$$

Now $d_2: E_2{}^{01} \to E_2{}^{20}$ sends $\tilde{z}$ to $\tilde{x}\tilde{y}$ since d_2 is given by the class of the group extension [6]. Therefore on E^{11} we have $d_2(\tilde{x} \otimes \tilde{z}) = \tilde{x}^2\tilde{y} \otimes 1 = 0$, so that $\tilde{x} \otimes \tilde{z}$ survives to E_3 and hence to E_∞. Since one can construct Steenrod operations (so in particular, the Bockstein) in the spectral sequence [8] we have

$$d_2(\beta(\tilde{x} \otimes \tilde{z})) = 0 \qquad d_3(\beta(\tilde{x} \otimes \tilde{z})) = 0$$

so that $\beta(\tilde{x} \otimes \tilde{z})$ survives to E_4 and hence to E_∞ where it is the mod p reduction of some integral class $s \in H^3(G, \mathbb{Z})$. Similarly we have a class $t \in H^3(G, \mathbb{Z})$ whose mod p reduction is $\beta(\tilde{y} \otimes \tilde{z})$.

We compute the action of automorphisms of G on s and t. If A is any automorphism it induces automorphisms A' of C and A'' of W. A' is specified by $z \to z^r$ and A'' is specified by some 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. For A' and A'' to fit together to give an automorphism of G we must have

$$A'[x, y] = [A''x, A''y]$$

or $r = ad - bc = \det A''$. Then by naturality we have

$$\begin{aligned} A^*\beta(\tilde{x} \otimes \tilde{z}) &= \beta[A''^*\tilde{x} \otimes A'^*\tilde{z}] \\ &= r\{a\beta(\tilde{x} \otimes \tilde{z}) + b\beta(\tilde{y} \otimes \tilde{z})\}. \end{aligned}$$

Similarly $A^*\beta(\tilde{y} \otimes \tilde{z}) = r\{c\beta(\tilde{x} \otimes \tilde{z}) + d\beta(\tilde{y} \otimes \tilde{z})\}$.
Then on $\Lambda^2 H^3(G; \mathbb{Z}/p)$ we have

$$\begin{aligned} A^*\beta\{(\tilde{x} \otimes \tilde{z}) \wedge \beta(\tilde{y} \otimes \tilde{z})\} &= r^2 \det A''\{\beta(\tilde{x} \otimes \tilde{z}) \wedge \beta(\tilde{y} \otimes \tilde{z})\} \\ &= r^3\{\beta(\tilde{x} \otimes \tilde{z}) \wedge \beta(\tilde{y} \otimes \tilde{z})\}. \end{aligned}$$

Now suppose for some integer l we have $A^*(s \wedge t) = \pm ls \wedge lt$ in $\Lambda^2 H^3(G; \mathbb{Z})$. Then taking mod p reductions we get

$$r^3\{\beta(\tilde{x} \otimes \tilde{z}) \wedge \beta(\tilde{y} \otimes \tilde{z})\} = \pm l^2\{\beta(\tilde{x} \otimes \tilde{z}) \wedge \beta(\tilde{y} \otimes \tilde{z})\} \quad \text{in} \quad \Lambda^2 H^3(G; \mathbb{Z}/p)$$

so

$$l^2 = (\pm r)^3 \bmod p$$

which proves the lemma.

III. PROOF OF THEOREM B

§1. APPROXIMATING *BG* BY A VARIETY

For any finite group G we build quasi-projective varieties which approximate BG up to arbitrarily large skeleta and compute the tangent bundles of the varieties.

Let G have order k and let $\pi\colon G \to \sum_k$ be the regular representation of G viewed as an injection into the symmetric group on k letters. For any integer n let V_n be the Stiefel manifold of ordered complex k-frames in n-space. Let G act on each frame by permuting the vectors: $g \cdot (v_1, \ldots, v_k) = (v_{\pi g(1)}, \ldots, v_{\pi g(k)})$. This defines a free G-action on V_n. If F is any free G-space and ρ is any complex representation of G we can form the vector bundle $F \times^G \mathbb{C}^{\dim \rho}$ over F/G. Denote the bundle by $F(\rho)$.

LEMMA III.1. *Let W_n be the quotient V_n/G. Then W_n has the homotopy type of BG up through dimension $2(n-k)$. The tangent bundle of W_n, $\tau(W_n)$, is $V_n(n\pi)$ where $n\pi$ is the kn-dimensional representation $\pi \oplus \pi \oplus \cdots \oplus \pi$ (n times).*

Proof. Let $\bar{V}_n = U(n)/U(n-k)$ be the Stiefel manifold of unitary k-frames in n-space.

Then $\bar{V}_n$ is $2(n-k)$-connected [14]. On the other hand V_n is homeomorphic to $\bar{V}_n \times R^{k^2}$ since every non-singular map $\mathbb{C}^k \xrightarrow{v} \mathbb{C}^n$ can be factored uniquely as $\mathbb{C}^k \xrightarrow{h} \mathbb{C}^k \xrightarrow{v} \mathbb{C}^n$ where v is norm-preserving (unitary) and h is given by a positive-definite hermitian matrix. (The standard "polar decomposition" proof, e.g. in [5] goes through without substantial change.) Hence V_n is also $2(n-k)$-connected and so W_n is homotopy equivalent to BG up to the $2(n-k)$ skeleton.

To determine the tangent bundle of W_n we note that V_n is the open subset of $\mathbb{C}^n \oplus \cdots \oplus \mathbb{C}^n$ (k times) consisting of vectors $v_1, \ldots, v_k$ with $v_1 \wedge \cdots \wedge v_k \neq 0$. Let $\dfrac{\partial}{\partial v_i}$ denote the standard basis of the tangent space of the ith copy of $\mathbb{C}^n$ at v_i. Then V_n is parallelizable and a basis for the tangent space at $v_1, \ldots, v_k$ is given by the frame $\dfrac{\partial}{\partial v_1}, \ldots, \dfrac{\partial}{\partial v_k}$. The G-action on the tangent bundle of V_n is

$$g \cdot \left(v_1, \ldots, v_k, \frac{\partial}{\partial v_1}, \ldots, \frac{\partial}{\partial v_k} = \right)\left(v_{\pi g(1)}, \ldots, v_{\pi g(k)}, \frac{\partial}{\partial v_{\pi g(1)}}, \ldots, \frac{\partial}{\partial v_{\pi g(k)}}\right)$$

so the quotient is $V_n \times^G [\mathbb{C}^n \oplus \cdots \oplus \mathbb{C}^n]$ where G acts on $\mathbb{C}^n \oplus \cdots \oplus \mathbb{C}^n$ by n copies of the regular representation. This is by definition the bundle $V_n(n\pi)$. Since G is a finite group the quotient of the tangent bundle of V_n is the tangent bundle of W_n.

§2. A CRITERION FOR DIFFEOMORPHISM

Given complex representations α and β of G we can form the bundles $V_n(\alpha)$ and $V_n(\beta)$ over W_n. We study when the total spaces of these bundles are diffeomorphic.

In general let S and T be vector bundles over a space X and suppose we have a diffeomorphism of the total spaces $\tilde{f}\colon |T| \to |S|$. Then, in particular, $\tilde{f}$ is a tangential homotopy equivalence, i.e. a homotopy equivalence with $\tilde{f}^*\tau|S| = \tau|T|$. On the other hand, the total space of, say, S is homotopy equivalent to X

$$\begin{tikzcd} {|S|} \arrow[r, bend left, "p"] & X \arrow[l, bend left, "s"] \end{tikzcd} \quad \text{with} \quad s^*\tau|S| = S \oplus \tau X$$

and similarly for T, so we can push $\tilde{f}$ down to a homotopy automorphism f of X satisfying

$$f^*(S \oplus \tau X) = T \oplus \tau X.$$

Now apply this to our variety W_n. Let α and β be two complex d-dimensional representations of G and form the bundles $V_n(\alpha)$ and $V_n(\beta)$. Then if these are diffeomorphic there must exist a homotopy automorphism f_n of W_n with

$$f_n^*(V_n(\alpha) \oplus \tau W_n) = V_n(\beta) \oplus \tau W_n \quad \text{or, by III.1,}$$

$$f_n^*(V_n(\alpha) \oplus V_n(n\pi)) = V_n(\beta) \oplus V_n(n\pi) \tag{1}$$

in the real K-theory of W_n.

Let EG be a contractible free G-space. Then for any representation ρ we form the bundle $E(\rho)$ over BG. Let $\mathscr{K}^*(BG)$ be the inverse limit of the complex K-theory of the n-skeleta of BG. Then we have

PROPOSITION III.2. *Suppose the total spaces $V_n(\alpha)$ and $V_n(\beta)$ are diffeomorphic for every n. Then there exists an automorphism A of G such that*

$$EG(A^*\alpha \oplus A^*\bar{\alpha}) = EG(\beta \oplus \bar{\beta})$$

in $\mathscr{K}^*(BG)$. Here $\bar{\alpha}$ and $\bar{\beta}$ denote the complex conjugates of α and β.

Proof. Suppose $V_n(\alpha)$ and $V_n(\beta)$ are diffeomorphic. Then the homotopy equivalence f_n in equation (1) restricts to a unique equivalence on the $2(n-k)-1$ skeleton of W_n which is equivalent to the $2(n-k)-1$ skeleton of BG. Thus we get a homotopy automorphism $\hat{f}_n$ of a skeleton of BG. By obstruction theory $\hat{f}_n$ is the restriction of some homotopy automorphism A_n of BG.

$$\begin{tikzcd}
W_n \arrow[r, "f_n"] & W_n \\
W_n^{(2(n-k)-1)} \arrow[u, hook, "j"] \arrow[r, "f'_n"] \arrow[d, "h"] & W_n^{(2(n-k)-1)} \arrow[u, hook, "j"'] \arrow[d, "h"'] \\
BG^{(2(n-k)-1)} \arrow[r, "\hat{f}_n"] \arrow[d, hook, "i"] & BG^{(2(n-k)-1)} \arrow[d, hook, "i"'] \\
BG \arrow[r, "A_n"] & BG
\end{tikzcd}$$

j is the inclusion of the $2(n-k)-1$ skeleton of W_n

h is the equivalence of skeleta

i is the inclusion of the $2(n-k)-1$-skeleton of BG

Now for any representation ρ of G, the bundle $V_n(\rho)$ over W_n pulls back from the bundle $EG(\rho)$ over BG, i.e. $h^*i^*EG(\rho) = j^*V_n(\rho)$ where h is the equivalence of the $2(n-k)-1$-skeleta of W_n and BG. So using equation (1) and referring to the diagram above we have

$$j^*f_n^*[V_n(\alpha) \oplus V_n(n\pi)] = j^*[V_n(\beta) \oplus V_n(n\pi)]$$

$$f_n'^*j^*[V_n(\alpha) \oplus V_n(n\pi)] = j^*[V_n(\beta) \oplus V_n(n\pi)]$$

$$f_n'^*h^*i^*[EG(\alpha) \oplus EG(n\pi)] = h^*i^*[EG(\beta) \oplus EG(n\pi)]$$

$$h^*\hat{f}_n^*i^*[EG(\alpha) \oplus EG(n\pi)] = h^*i^*[EG(\beta) \oplus EG(n\pi)]$$

then since h is a homotopy equivalence

$$\hat{f}_n^*i^*[EG(\alpha) \oplus EG(n\pi) = i^*[EG(\beta) \oplus EG(n\pi)]$$

or

$$i^*A_n^*[EG(\alpha) \oplus EG(n\pi)] = i^*[EG(\beta) \oplus EG(n\pi)]$$

as real vector bundles over the $2(n-k)-1$ skeleton of BG.

Now A_n is an automorphism of G and so $A_n^*EG(n\pi) = EG(n\pi)$ as bundles over BG

because the regular representation is invariant under automorphisms of G. Thus we have $i^*A_n{}^*EG(\alpha) = i^*EG(\beta)$ in the real K-theory of the $2(n-k)-1$ skeleton of BG. But there are only a finite number of automorphisms of G. So if $V_n(\alpha)$ and $V_n(\beta)$ are diffeomorphic for *every* n there is some particular automorphism A so that $i^*A^*EG(\alpha) = i^*EG(\beta)$ where i^* is the restriction to arbitrarily large skeleta of BG. That is

$$A^*EG(\alpha) = EG(\beta) \quad \text{in} \quad \mathscr{K}_{\mathbb{R}}{}^*(BG) \tag{2}$$

where $\mathscr{K}_{\mathbb{R}}{}^*(BG)$ stands for the inverse limit of the real K-theory of the n-skeleta.

To restate this condition in terms of complex K-theory we use the maps $c\colon K_{\mathbb{R}} \to K$ and $r\colon K \to K_{\mathbb{R}}$, complexification and realification. Then for any complex vector bundle S, $\mathrm{cr}\, S = S \oplus \bar{S}$. So apply cr to both sides of equation (2) and note that, for a representation ρ, $\overline{EG(\rho)} = EG(\bar{\rho})$.

§3. PROOF OF THE THEOREM

Theorem B now follows easily from III.2. First of all, V_n is a quasi-projective variety defined over $\mathbb{Q}$. For any representation α, $V_n(\alpha)$ is also quasi-projective, since it is a finite quotient of $V_n \times \mathbb{C}^N$, and defined over $\mathbb{Q}(\omega)$ if α is defined over $\mathbb{Q}(\omega)$. Since the G-action on V_n is defined over $\mathbb{Q}$ this also shows that $V(\alpha^\sigma) = [V(\alpha)]^\sigma$.

Suppose now that $V_n(\alpha)$ and $V_n(\alpha^\sigma)$ are diffeomorphic for every n. Then III.2 implies that there is an automorphism A of G with

$$EG(A^*(\alpha \oplus \bar{\alpha})) = EG(\alpha^\sigma \oplus \bar{\alpha}^\sigma)$$

in $\mathscr{K}^*(BG)$. But if G is a p-group we have Atiyah's result [2] that the map $\rho \to EG(\rho)$ from the representation ring of G to $\mathscr{K}^*(BG)$ is injective. Hence

$$A^*(\alpha \oplus \bar{\alpha}) = \alpha^\sigma \oplus \bar{\alpha}^\sigma$$

in the representation ring $R(G)$. But this is precisely what is ruled out by Condition **B**.

Q.E.D

We wish to prove the remark about homeomorphism made in I.3. For this we use the theory K_{TOP} of topological vector bundles.

There is a natural map from $K_{\mathbb{R}}$ to K_{TOP}. Let S be the set of regular primes p such that 2 has even order in the group of units $(\mathbb{Z}/p)^*$. Then for $p \in S$ it is a result of Adams and Sullivan [15] that there is an injection of profinite cohomology theories

$$(K_{\mathbb{R}}{}^*)_p^\wedge \to (K_{\mathrm{TOP}}^*)_p^\wedge .$$

On the other hand, Atiyah's results [2] imply that

$$\mathscr{K}_{\mathbb{R}}{}^*(BG) = K_{\mathbb{R}}{}^*(BG)_p^\wedge$$

for G any p-group. Thus we see that

$$\mathscr{K}_{\mathbb{R}}{}^*(BG) \to \mathscr{K}_{\mathrm{TOP}}^*(BG)$$

injects for p-groups with $p \in S$. This allows us to prove

COROLLARY III.3. *Let G be a p-group, $p \in S$, and α, σ chosen to satisfy Condition B. Then, for some n, $V_n(\alpha)$ and $V_n(\alpha^\sigma)$ are not homeomorphic.*

Proof. Suppose $V_n(\alpha)$ and $V_n(\beta)$ are homeomorphic for each n. If $f: M \to N$ is a homeomorphism of two manifolds then $f^*\tau(N) = \tau(M)$ as topological bundles over M. Using this fact, the same argument as in III.2 produces an automorphism A such that equation (2) of III.2 holds in $\mathscr{K}_{\mathrm{TOP}}(BG)$, where $\mathscr{K}_{\mathrm{TOP}}$ is the inverse limit of K_{TOP} of the finite skeleta of BG. But $\mathscr{K}_{\mathbb{R}}(BG) \to \mathscr{K}_{\mathrm{TOP}}(BG)$ is an injection. Thus equation (2) of III.2 actually holds in $\mathscr{K}_{\mathbb{R}}(BG)$ so we can complete the proof exactly as before.

Question. Is it true that $\mathscr{K}_{\mathbb{R}}(BG) \to \mathscr{K}_{\mathrm{TOP}}(BG)$ injects for *any* p-group (even without $p \in S$)? If so, then Theorem B will always provide non-homeomorphic examples.

§4. EXAMPLES

In this section we find some p-groups which satisfy Condition B and hence can be the fundamental groups of non-diffeomorphic varieties. We first replace Condition B by a more "group theoretic" condition. Given a map (not an automorphism) $\phi: G \to G$ we say that ϕ is *conjugate to an automorphism* of G if there exists an automorphism $B: G \to G$ such that for all $g \in G$, $B(g)$ and $\phi(G)$ are in the same conjugacy class.

LEMMA III.4.1. *Let G be a finite group, ω a $|G|$th root of unity and $\sigma \in \mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})$ with $\sigma(\omega) = \omega^l$. Then there always exists some representation α of G with G, σ, α satisfying Condition B unless the map $g \to g^{l^2}$ is conjugate to an automorphism of G.*

Proof. Let $\rho_1, \dots, \rho_s$ be the inequivalent irreducible representations of G. Let

$$m: \{1, \dots, s\} \to \mathbb{Z}$$

be any "indexing function" with the property that

$$m(i) + m(j) = m(i') + m(j')$$

implies either $i = i', j = j'$ or $i = j', j = i'$. (For example, take $m(i) = 3^i$.) Let α be the representation

$$\alpha = \sum_{i=1}^{s} m(i)\rho_i .$$

Now, if G, σ, α does not satisfy Condition B there exists an automorphism A with

$$\sum_{i=1}^{s} m(i)\{A^*\rho_i \oplus A^*\bar{\rho}_i\} = \sum_{i=1}^{s} m(i)\{\rho_i \oplus \bar{\rho}_i{}^\sigma\}.$$

Let $c: \{1, \dots, n\} \to \{1, \dots, n\}$ be given by $\bar{\rho}_i = \rho_{c(i)}$ (we could have $i = c(i)$ for some i). Then, since $cc(i) = i$ we can rewrite the above as

$$\sum_{i=1}^{s} [m(i) + m(c(i))]A^*(\rho_i) = \sum_{i=1}^{s} [m(i) + m(c(i))]\rho_i{}^\sigma.$$

By choice of m, $A^*(\rho_i)$ and $A^*(\rho_{c(i)})$ are the only irreducible representations appearing in the sum on the left with multiplicity precisely $m(i) + m(c(i))$. Similarly $\rho_i{}^\sigma$ and $\rho^\sigma_{c(i)}$ are the only irreducible representations with that multiplicity in the sum on the right. Since irreducible decomposition is unique we see that for every irreducible representation ρ_i either $A^*(\rho_i) = \rho_i^\sigma$ or $A^*(\rho_i) = \rho^\sigma_{c(i)} = \bar{\rho}_i^\sigma$. Since when we take characters we have $\chi\rho^\sigma(g) = \chi\rho(g^l)$ for all $g \in G$, we see that

either $$\chi\rho_i(Ag) = \chi\rho_i(g^l) \quad \text{for all} \quad g \in G$$

or $$\chi\rho_i(Ag) = \chi\bar{\rho}_i(g^l) = \chi\rho_i(g^{-l}) \quad \text{for all} \quad g \in G.$$

Now let B be the automorphism $A \circ A$. Then for all $g \in G$ we have

$$\begin{aligned}\chi\rho_i(Bg) &= \chi\rho_i(AAg) = \chi\rho_i((Ag)^{\pm l}) = \chi\rho_i(Ag^{\pm l}) \\ &= \chi\rho_i(g^{l^2})\end{aligned}$$

for every irreducible representation ρ_i. Hence Bg and g^{l^2} are conjugate in G.

Q.E.D.

Using this result we can generate p-groups satisfying Condition B. In fact, there are p-groups which have the even stronger

Condition C. The map $g \to g^k$ is not conjugate to an automorphism of G unless $k \equiv 1 \pmod p$. For such groups we have

COROLLARY III.4.2. *Let G be a p-group satisfying Condition C and σ* any *field automorphism of $\mathbb{C}$. Then there exist homotopy equivalent non-diffeomorphic conjugate varieties V and V^σ with fundamental group G, as long as σ restricted to $\mathbb{Q}$(p*th *roots of unity) is not the identity or complex conjugation.*

Proof. Let ω be a primitive $|G|$th root of unity and suppose $\sigma(\omega) = \omega^l$. Then, if we cannot build such a variety V, III.4.1 and Theorem B imply that $l^2 = \pm 1 \pmod p$ and so $\sigma(e^{2\pi i/p}) = e^{\pm 2i\pi/p}$.

Q.E.D.

Example 1. Let G be a central extension of $\mathbb{Z}/p$ by a $\mathbb{Z}/p$ vector space W.

$$0 \to \mathbb{Z}/p \to G \to W \to 0.$$

Then the extension determines a bilinear form $\Gamma\colon W \times W \to \mathbb{Z}/p$ as follows: Pick an additive generator z for $\mathbb{Z}/p$. For each $w \in W$ pick $\tilde{w} \in G$ projecting to W. Then set $\Gamma(v, w) = r$ where $[\tilde{v}, \tilde{w}] = rz$. Then if Γ is not identically zero (i.e. G is nonabelian) any automorphism $\mathrm{B}\colon G \to G$ induces automorphisms B' of $\mathbb{Z}/p$ and B'' of W. If B is conjugate to $g \to g^k$ the induced automorphisms must in fact be $B'\colon z \to kz$ and $B''\colon w \to kw$ since these groups are abelian. But any automorphism must preserve the bilinear form:

$$B'\Gamma(v, w) = \Gamma(B''v, B''w)$$

or $$k\Gamma(v, w) = \Gamma(kv, kw) = k^2\Gamma(v, w)$$

which implies $k = 1 \bmod p$ as long as $\Gamma(v, w) \neq 0$ for some v, w.

Example 2. (This example was kindly communicated to the author by George Bergman of the University of California at Berkeley.)

Let G be the semidirect product of $\mathbb{Z}/p$ and $\mathbb{Z}/p^2$ (p odd) defined by

$$x^p = 1 \qquad y^{p^2} = 1 \qquad yx = xy^{1+p}$$

(note that $(1+p)^i \equiv 1 + pi \pmod{p^2}$). Then one can verify that for any i and j $x^{-1}(x^i y^j)x = (x^i y^j)^{1+p}$. (Use the fact that $(x^i y^j)^p = y^{jp}$.)

Now let ϕ be any automorphism of G. Then $\phi(y)$ is some element of order p^2. Suppose $\phi(x) = x^k$. Then

$$\phi(y)^{1+p} = \phi(x^{-1}yx) = x^{-k}\phi(y)x^k = \phi(y)^{1+kp}$$

so that $k \equiv 1 \pmod p$. Taking account of inner automorphisms, we see that G satisfies Condition C.

Notice that Examples 1 and 2 work for any odd prime, so that we also get non-homeomorphic varieties by III.3.

REFERENCES

1. M. Artin and B. Mazur: *Étale Homotopy*. Lecture Notes in Mathematics, Vol. 100. Springer, Berlin (1969).
2. M. F. Atiyah: *Characters and Cohomology of Finite Groups*. I.H.E.S. Publications No. 9 (1961).
3. M. F. Atiyah and F. Hirzebruch: Analytic cycles on complex manifolds, *Topology* **1** (1962), 25–45.
4. R. Brauer: On the representations of a group G in the field of Gth roots of unity, *Am. J. Math.* **67** (1945), 461–471.
5. C. Chevalley: *Theory of Lie Groups*, Vol I. Princeton (1946).
6. G. Hochschild and J. P. Serre: Cohomology of group extensions, *Trans. Am. math. Soc.* (1953).
7. W. C. Hsiang: Higher obstructions to sectioning a certain type of fibre bundles, *Trans. Am. math. Soc.* (1964), 393–412.
8. L. Kristensen: On the cohomology of two-stage Postnikov systems, *Acta Math.* **107** (1962), 73–123.
9. E. G. Kundert: Über Schnittflächen in Speziellen Faserungen und Felder reeler und komplexer Linienelemente, *Ann. Math.* **54** (1951), 215–246.
10. I. Schur: Über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen, *J. Reine Angew. Math.* **127** (1904), 20–50.
11. J. P. Serre: Géométrie algébrique et géométrie analytique, *Ann. Inst. Fourier* **6** (1955), 1–42.
12. J. P. Serre: Exemples de variétés projective conjugées non homémorphes, *C. r. Acad. Sci. Paris* **258** (1964), 4194–4196.
13. J. P. Serre: Sur la topologie des variétés algebriques en characteristique p, *Symposium Internacional de Topologia Algebraica*, pp. 24–53. Mexico (1958).
14. N. E. Steenrod: *The Topology of Fibre Bundles*. Princeton (1951).
15. D. Sullivan: *Localizaton, Periodicity, and Galois Symmetry* (mimeo). M. I. T. (1970).

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